ON THE GENERALIZED SCATTERING MATRIX OF A LOSSLESS MULTIPORT

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ABSTRACT

In characterizing interacting discontinuities in waveguide increasing use is made of the so called generalized scattering matrix (GSM) formalism in order to describe 'accessible modes' below cutoff. This is motivated by the intrinsic numerical stability of the scattering matrix with respect to the transmission matrix, for istance. Unfortunately, we found that in many recent works and textbooks the GSM of a length of waveguide involving a mode above cutoff and several ones below cutoff is defined so that unitarity of the matrix and hence power conservation is not preserved.
Still, numerical values of the fundamental mode scattering parameters reported as examples by the same references appear to be correct.
In solving this apparent paradox, we address the problem of correctly defining the GSM for lossless modes below cutoff and alert the reader to the drawbacks of using a definition that does not maintain unitarity.

INTRODUCTION

Modern analysis of microwave components with the help of powerful computers makes massive use of the computation of the generalised scattering matrices of discontinuities interacting via several modes of which the fundamental is normally in propagation while the rest is below cutoff but still causing interaction with adjacent discontinuities (accessible modes [1]).
A fundamental building block of the GSM analysis consists of the GSM of a length of waveguide interconnecting two successive discontinuities.
From circuit point of view this is equivalent to a set of parallel uncoupled transmission lines where the line corresponding to the fundamental mode has real characteristic impedance Z_0 and propagation constant β, whereas the remaining lines have pure imaginary characteristic impedance jX_0 and real attenuation factor α.
As will be shown in the following, the form of the scattering matrix of a line below cutoff that is often currently assumed in literature is defined in such a manner that it does not satisfy the condition of unitarity as is imposed by power conservation (see [2], [3], [4], [5] among many others).

We will consider the problem of correctly defining the scattering matrix and present a solution that is seen to satisfy all physical requirements.

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THE CURRENT DEFINITION OF S MATRIX OF A LENGTH OF WAVEGUIDE

The form of the scattering matrix of a length of waveguide is often taken as

$$\begin{bmatrix} 0 & e^{-j\beta l} \\ e^{-j\beta l} & 0 \end{bmatrix} \text{ for a propagating mode} \quad (1)$$

$$\begin{bmatrix} 0 & e^{-\alpha l} \\ e^{-\alpha l} & 0 \end{bmatrix} \text{ for a non propagating mode} \quad (2)$$

Where β is the propagation constant of the mode above cutoff and α the attenuation of the mode below cutoff. Since there is no physical definition based on wave amplitude of the scattering matrix of a mode below cutoff, (2) is arrived at by setting $\alpha = -j\beta$ in (1). Since both forms portray to represent the SM of a lossless circuit the unitary condition $\mathbf{S}^{+}\mathbf{S} = \mathbf{I}$ ought to be satisfied. It is easy to check however that only the first form for real

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β satisfies unitarity whilst the second one gives

$$\mathbf{S}^{+}\mathbf{S} = \begin{bmatrix} e^{-2\alpha l} & 0 \\ 0 & e^{-2\alpha l} \end{bmatrix} \quad (3)$$

which is obviously not the unit matrix.

APPROPRIATE DEFINITION OF THE GSM

Going back to the principles of network theory, we find that the scattering matrix is formally defined for any n-port network as [6], [7]

$$\mathbf{S} = (\mathbf{Z}-\mathbf{I})(\mathbf{Z}+\mathbf{I})^{-1} = (\mathbf{Z}+\mathbf{I})^{-1}(\mathbf{Z}-\mathbf{I}) \quad (4)$$

Where the adimensional impedance matrix $\mathbf{Z}$ is supposed to exist, or, alternatively, if the normalized admittance matrix $\mathbf{Y}$ exists:

$$\mathbf{S} = (\mathbf{I}-\mathbf{Y})^{-1}(\mathbf{I}+\mathbf{Y}) \quad (5)$$

$\mathbf{S}$ links the wave amplitudes $\mathbf{a}$, $\mathbf{b}$ in such a way that

$$\mathbf{b} = \mathbf{S}\mathbf{a} \quad (6)$$

$\mathbf{a}$ and $\mathbf{b}$ are defined in term of "appropriately normalized" voltages and currents as:

$$\mathbf{a} = \frac{1}{2}(\mathbf{v}+\mathbf{i}) \quad (7)$$

$$\mathbf{b} = \frac{1}{2}(\mathbf{v}-\mathbf{i}) \quad (8)$$

where the normalized quantities $\mathbf{v}$ and $\mathbf{i}$ are related by $\mathbf{v} = \mathbf{Z}\mathbf{i}$. It is very easy to check that definition (4) satisfies unitarity. For, if the network is reciprocal and lossless, $\mathbf{Z} = -\mathbf{Z}^{+}$ so that by substituting (4) in the unitarity condition, we obtain

$$\mathbf{S}^{+}\mathbf{S} = (\mathbf{Z}^{+}+\mathbf{I})^{-1}(\mathbf{Z}^{+}-\mathbf{I})(\mathbf{Z}-\mathbf{I})(\mathbf{Z}+\mathbf{I})^{-1} =$$
$$= (\mathbf{Z}-\mathbf{I})^{-1}(\mathbf{Z}+\mathbf{I})(\mathbf{Z}-\mathbf{I})(\mathbf{Z}+\mathbf{I})^{-1} =$$
because of commutativity
$$= (\mathbf{Z}-\mathbf{I})^{-1}(\mathbf{Z}+\mathbf{I})(\mathbf{Z}+\mathbf{I})^{-1}(\mathbf{Z}-\mathbf{I}) = \mathbf{I} \quad (9)$$

Let us consider how the above definition applies to the case of a line below cutoff of characteristic impedance jX_0 with real X_0 and attenuation $\xi = \alpha l$.

Starting from the well known form of the Z-matrix of this line,

$$\mathbf{Z} = -jX_0 \begin{bmatrix} \coth\xi & 1/\sinh\xi \\ 1/\sinh\xi & \coth\xi \end{bmatrix} \quad (10)$$

Upon application of (4) we obtain

$$\mathbf{S} = \frac{1}{1 - X_0^2 + 2jX_0\coth\xi} \begin{bmatrix} -(1+X_0^2) & \frac{2jX_0}{\sinh\xi} \\ \frac{2jX_0}{\sinh\xi}\xi & -(1+X_0^2) \end{bmatrix} \quad (11)$$

We note that:

1. This matrix is unitary ;
2. The S_{11} and S_{22} parameters are different from zero, compatibly with the fact that a line below cutoff reflects power;
3. As ξ tends to infinity, S_{11} and S_{22} tend to 1 whereas S_{12} tends to zero;
4. As ξ tends to zero, S_{11} and S_{22} tend to 0 whereas S_{12} tends to 1;
5. in the limiting case of a mode above cutoff $X_0 = -j$, $\xi = j\beta l$ and the standard form (1) is recovered.

Let us now consider the question of normalization in the multimodal case.

In the multiple line case the normalized voltages and currents are normally defined as

$$\mathbf{v} = \mathbf{R}^{-1/2}\mathbf{V} = \frac{1}{2}(\mathbf{a}+\mathbf{b}) \quad (12)$$

$$\mathbf{i} = \mathbf{R}^{+1/2}\mathbf{I} = \frac{1}{2}(\mathbf{a}-\mathbf{b}) \quad (13)$$

Where $\mathbf{V}$ and $\mathbf{I}$ are the actual unnormalized voltages and currents and the diagonal dimensional matrix $\mathbf{R}$ is to be chosen as real for power to be conserved. In fact, if $\mathbf{R}$ is not real, then $\mathbf{v}^{+}\cdot\mathbf{i} = \mathbf{V}^{+}(\mathbf{R}^{+})^{-1/2}\cdot\mathbf{R}^{1/2}\mathbf{I}$ so that $\mathrm{Re}\{\mathbf{v}^{+}\cdot\mathbf{i}\} \neq \mathrm{Re}\{\mathbf{V}^{+}\cdot\mathbf{I}\}$ and power is not conserved.

The often encountered definition $R_{kk} = \mathrm{Re}\{Z_{0k}\}$ clearly fails in the case we are considering, Z_k being pure imaginary for a mode below cutoff.

It is noted that where (2) is used, the normalization is usually carried out with respect to an imaginary impedance jX_0, contrary to the principle just illustrated. In practice, however, only the block of the S-matrix corresponding to propagating modes is retained in computing the input-output quantities and this conceptually incorrect normalization works out in such a way that the final result is still correct. In the latter, however, the useful check based on the unitarity of the GSM has been lost. In our definition instead is expedient to carry out the impedance normalization by assuming $R_{kk} = |Z_{0k}|$. Consequently, the normalized current impedance of all lines below cutoff results to be $sign(X_0)\,j$, while that of the lines above cutoff is 1, according to standard convention. In that case the 2×2 block corresponding to a line takes the form

$$\mathbf{d}_1 = \begin{bmatrix} 0 & e^{-j\beta l} \\ e^{-j\beta l} & 0 \end{bmatrix} \qquad (14)$$

for the fundamental mode.

$$\begin{aligned} \mathbf{d}_k &= \begin{bmatrix} j sign(X_{0k}) \tanh \xi_k & 1/\cosh \xi_k \\ 1/\cosh \xi_k & j sign(X_{0k}) \tanh \xi_k \end{bmatrix} \\ k &= 2, 3.. \end{aligned} \qquad (15)$$

for modes below cutoff.

The overall GSM of a length of waveguide with a single propagating mode takes the form:

$$\mathbf{S} = \begin{bmatrix} \mathbf{0} & \mathbf{D} \\ \mathbf{D} & \mathbf{0} \end{bmatrix}$$
$$\text{where } \mathbf{D} = \begin{bmatrix} \mathbf{d}_1 & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{d}_2 & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{d}_N \end{bmatrix} \qquad (16)$$

Of course, the above normalization applies to the GSM of any linear device characterized by an impedance matrix **Z**.

CLOSURE OF THE S MATRIX AT THE INPUT AND OUTPUT PORT

Since the input and output ports of the device are considered to be terminated by infinite lengths of waveguide, all ports corresponding to higher order modes have to be closed on their characteristic impedance, that is $j\,sign(X_{0k})$ according to the normalization assumed in (16).

Therefore ,the reduction of the GSM to the ordinary $2n_a \times 2n_a$ scattering matrix $\mathbf{s}$, where n_a is the number of modes *above* cutoff, takes place by using the standard port reduction formula

$$\mathbf{s} = \mathbf{S}_{aa} - \mathbf{S}_{ab}(\mathbf{J} - \mathbf{S}_{bb})^{-1}\mathbf{S}_{ba} \qquad (17)$$

Where $\mathbf{J}$ is a $(2n_b \times 2n_b)$ diagonal matrix, n_b being the number of *accessible* modes *below* cutoff, whose k-th element is given by:

$$[\mathbf{J}]_{kk} = [\mathbf{J}]_{k+n_b,k+n_b} = j\,sign(X_{0k}) \;\; k = 1, n_b \qquad (18)$$

block $\mathbf{S}_{ab}$ relates accessible modes of type 'a', above cutoff, to those of type 'b', below cutoff. Although the proposed definition of GSM is a little more time consuming with respect to the standard one, as it requires reduction, nonetheless it provides a useful criterium for checking algebraic implementation through its unitarity.

USEFULNESS OF THE NEW DEFINITION OF GSM

As previously observed, the main advantage of the new definition of the GSM is its unitarity, since the latter constitutes a significant test for checking the correctness of the numerical results. On the contrary, when using the standard definition one can check only the unitarity of the block relating modes above cutoff, losing all information on the blocks involving accessible modes below cutoff. A simple example can illustrate a situation where the unitarity of the GSM permits to uncover an error otherwise hard to detect.

Let us consider the waveguide Y-junction (H-plane) shown in fig.1. Be $\mathbf{S}_{nm}^{IJ}$ the transmission between the n-th mode of the I-th port and the m-th mode of the J-th port. If the n-th and the m-th modes have the same parity, then $\mathbf{S}_{nm}^{21} = \mathbf{S}_{nm}^{31}$, otherwise $\mathbf{S}_{nm}^{21} = -\mathbf{S}_{nm}^{31}$. Now, suppose to make the trivial, but very insidious mistake, consisting of setting $\mathbf{S}_{nm}^{21} = \mathbf{S}_{nm}^{31}$, $\forall\, n, m$. It is immediate to observe that the block of the GSM relative to modes above cutoff, as normally defined, continues to satisfy unitarity. Therefore an inspection of that property fails to detect the error. On the contrary, by checking the unitarity of the GSM as correctly defined, the mistake emerges immediately. This is just one of many simple and realistic examples showing the usefulness of the correct definition of the GSM.

CONCLUSIONS

In dealing with higher order mode interaction between successive discontinuities it is important to properly define the GSM.
We give a correct definition of the GSM and show its usefulness by considering an example where unitarity permits to uncover an error that could possibly occurr from an incorrect implementation of the GSM of a Y-junction.

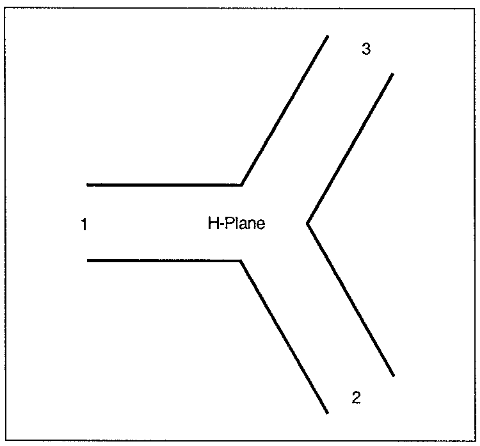

Fig. 1. H-plane section of the Y-junction considered in the example

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